

ON MEROMORPHIC FUNCTIONS WHICH ARE BRODY CURVES

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ABSTRACT. We discuss meromorphic functions on the complex plane which are Brody curves regarded as holomorphic maps to \mathbb{P}_1 , i.e., which have bounded spherical derivative.

We discuss meromorphic functions on \mathbb{C} which are Brody curves regarded as holomorphic curves from \mathbb{C} to $\mathbb{P}_1(\mathbb{C})$. In other words, those meromorphic functions for which $||f'||$ calculated with respect to the euclidean metric on \mathbb{C} and the Fubini-Study-metric on \mathbb{P}_1 is bounded.

In concrete terms this means:

$$\limsup \frac{|f'(z)|}{1 + |f(z)|^2} < +\infty.$$

This number is also called “spherical derivative” ([2]). We follow the established notation and denote this spherical derivative as

$$f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

Brody curves in general have been studied since the seminal paper of Brody ([1]) illuminated their relevance for hyperbolicity questions.

The “spherical derivative” $f^\#$ has been studied before Brody proved his theorem, especially by Clunie, Hayman, Lehto and Virtanen ([2],[3],[4]).

In particular, Clunie and Hayman showed that a meromorphic Brody function is of order at most two and a Brody entire function is of order at most one. They also demonstrated the existence of Brody curves with certain divisors with high multiplicities as prescribed zero divisors.

Easy examples for Brody functions are polynomials and the exponential function.

As noted by Tsukamoto ([5]), elliptic functions are also Brody: If Γ is a lattice in \mathbb{C} , and f is a Γ -periodic meromorphic function, then $f^\#$ is continuous and Γ -invariant. Therefore the existence of a compact fundamental region for Γ implies that $f^\#$ is bounded.

Here we will show that Brody functions remain Brody after multiplying with certain rational functions. We investigate two special classes of meromorphic functions, namely $R(z)e^z + Q(z)$ where R, Q

are rational functions and $e^z + e^{\lambda z}$ with $\lambda \in \mathbb{C}$. For these functions we determine completely which ones are Brody.

This provides us with some surprising examples: If f, g are Brody neither $f + g$, nor fg need to be Brody. The Brody condition is not closed: The meromorphic function $f_t(z) = e^z + \frac{z}{zt+1}$ is Brody if and only if $t \neq 0$.

In particular in view of the results of Clunie and Hayman on the order it might be natural to assume that every divisor of sufficiently slow growth can be realized as the zero divisor of an entire Brody function. We show that this is not the case, there are effective divisors of arbitrary slow growth which can not be realized as zero divisors of a Brody function.¹

1. BASIC PROPERTIES

We start with the observation

$$f^\# = \left| \frac{f'}{f} \right| \cdot h(f)$$

where

$$h(w) = \frac{|w|}{1 + |w|^2}.$$

Since $h(z) \leq \frac{1}{2}$ for all $z \in \mathbb{C}$, we may deduce: *If f'/f is bounded, then f is Brody.* In fact one has the following slightly stronger statement:

Lemma 1. *Let f be an entire function with*

$$\limsup_{z \rightarrow \infty} \left| \frac{f'(z)}{f(z)} \right| < +\infty.$$

Then f is Brody.

Proof. There is a compact subset $K \subset \mathbb{C}$ and a constant $C > 0$ such that $|f(z)/f'(z)| < C$ for all $z \notin K$. Then

$$\sup_{z \in \mathbb{C}} f^\#(z) \leq \max \left\{ \max_{z \in K} f^\#(z), C \right\}$$

□

Corollary 1. *The exponential function $z \mapsto e^z$ is Brody.*

Concerning rational functions the Brody property follows from the observation below:

Lemma 2. *Let R be a non-constant rational function on \mathbb{C} .*

Then $\lim_{z \rightarrow \infty} \frac{R'(z)}{R(z)} = 0$.

¹A related result has been obtained much earlier by Lehto ([3]).

Proof. Write R as a quotient of polynomials: $R = P/Q$. Then

$$\frac{R'}{R} = \frac{P'Q - PQ'}{PQ}$$

and the assertion follows from the fact that $\deg(P'Q - PQ') < \deg(PQ)$. \square

Corollary 2. *Rational functions have the Brody property.*

Proposition 1. *Assume that f is a Brody entire function, that R is a rational function and that $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$.*

Then $z \mapsto F(z) = R(f(\alpha z + \beta))$ is Brody, too.

Proof. Each rational function R defines a continuous self-map of \mathbb{P}_1 . The compactness of \mathbb{P}_1 implies that for each rational function R there exists a constant C such that the operator norm of the differential map $DR : T\mathbb{P}_1 \rightarrow T\mathbb{P}_1$ is bounded by C . Then $\sup F^\# \leq C|\alpha| \sup f^\#$. \square

Corollary 3. *The trigonometric functions \sin , \cos , \tan , \sinh , \cosh .. are all Brody.*

Proof. These functions can be expressed in the form $R(e^z)$ or $R(e^{iz})$ for some rational function $R \in \mathbb{C}(X)$. \square

2. PRODUCTS WITH RATIONAL FUNCTIONS

Lemma 3. *Let $f : \mathbb{C} \rightarrow \mathbb{P}_1$ be Brody and let R be a rational function with $R(\infty) \notin \{0, \infty\}$.*

Then $g = Rf$ is Brody.

Proof. We have

$$g^\#(z) = \frac{g'(z)}{g(z)} h(g(z))$$

with $h(w) = \frac{|w|}{1+|w|^2}$.

We will need the following auxiliary fact:

Claim: Let $\lambda > 1$. Then $h(\lambda w) \leq \lambda h(w)$ and $h(\lambda^{-1}w) \leq \lambda w$ for all $w \in \mathbb{C}$.

This claim is rather immediate:

$$h(\lambda w) = \frac{|\lambda w|}{1 + |\lambda w|^2} \leq \frac{|\lambda w|}{1 + |w|^2} = \lambda h(|w|)$$

and similarly for the second inequality.

Now write the rational function R as quotient of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)}.$$

Then

$$\frac{g'}{g} = \frac{f'}{f} + \frac{P'}{P} - \frac{Q'}{Q}$$

and (using $\lim(P'/P) = \lim(Q'/Q) = 0$) we may deduce that there are constants $R_1, C_1 > 0$ such that

$$\left| \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right| \leq C_1$$

for complex numbers z with $|z| > R_1$.

By assumption $\lim_{z \rightarrow \infty} R(z) \in \mathbb{C}^*$. Therefore there are constants $\lambda > 1$ and $R_2 > 0$ such that

$$\lambda^{-1} < |R(z)| < \lambda$$

whenever $|z| > R_2$.

Then

$$h(g(z)) = h(R(z)f(z)) \leq \lambda h(f(z))$$

if $|z| > R_2$.

As a consequence, we have

$$\begin{aligned} g^\#(z) &= \frac{|g'(z)|}{|g(z)|} h(g(z)) \leq \left(\frac{|f'(z)|}{|f(z)|} + C_1 \right) \lambda h(f(z)) \\ &\leq \lambda f^\#(z) + C_1 \lambda h(f(z)) \\ &\leq \lambda \left(f^\#(z) + \frac{C_1}{2} \right) \end{aligned}$$

if $|z| > \max\{R_1, R_2\}$.

Hence the assertion. \square

It is really necessary to assume $R(\infty) \neq 0, \infty$. For instance, consider the functions $f_1(z) = e^z + 1$ and $f_2(z) = 1/f_1(z)$. Both are Brody functions, but proposition 2 in the next section implies that neither $g_1(z) = zf_1(z)$ nor $g_2(z) = \frac{1}{z}f_2(z)$ is Brody.

3. CASE STUDY: $R(z)e^z + Q(z)$

We will now discuss certain sums.

Proposition 2. *Let R, Q be rational functions. Then $f(z) = R(z)e^z + Q(z)$ is a Brody function if and only if one of the following conditions is fulfilled:*

- (1) $R \equiv 0$ or
- (2) $Q(\infty) \neq \infty$.

Proof. If $R \equiv 0$, then $f = Q$ is rational and therefore Brody.

Assume now that $Q(\infty) \neq \infty$ and $R \not\equiv 0$. Then there is a rational function Q_0 with $Q_0(\infty) \notin \{0, \infty\}$ and a complex number c with $Q(z) = Q_0(z) + c$. Then

$$f(z) = Q_0(z) \left(\frac{R(z)}{Q_0(z)} e^z + 1 \right) + c$$

and f is Brody if and only if $g(z) = \frac{R(z)}{Q_0(z)} e^z$ is Brody. Now $S = R/Q_0$ is rational and $S \not\equiv 0$. Therefore

$$\lim_{z \rightarrow \infty} \frac{g'(z)}{g(z)} = \lim_{z \rightarrow \infty} \frac{S(z) + S'(z)}{S(z)} = 1.$$

But this implies

$$\limsup_{z \rightarrow \infty} g^\#(z) \leq 1 \cdot \frac{1}{2} < +\infty.$$

Hence g is Brody and therefore f is Brody, too.

It remains to show that f is not Brody whenever $Q(\infty) = \infty$ and $R \not\equiv 0$. In this case $Q(z) = Q_0(z)z^n$ for some rational function Q_0 with $Q_0(\infty) \notin \{0, \infty\}$ and $n \in \mathbb{N}$. Define $S = R/Q_0$. Then S is rational. Considering the equality

$$f(z) = Q_0(z) (S(z)e^z + z^n)$$

is Brody if and only if $g(z) = S(z)e^z + z^n$ is Brody. Since $R \not\equiv 0$ together with $Q_0(\infty) \neq 0$ implies $S \not\equiv 0$, the entire function g is transcendental. Therefore there is a sequence of complex numbers z_k with $\lim_{k \rightarrow \infty} |z_k| = \infty$ and $\lim_{k \rightarrow \infty} g(z_k) = 0$. Now

$$g'(z) = (S(z) + S'(z))e^z + nz^{n-1} = (S(z) + S'(z)) \frac{g(z) - z^n}{S(z)} + nz^{n-1}$$

which implies

$$g'(z) = z^n \left(\frac{S(z) + S'(z)}{S(z)} \left(\frac{g(z)}{z^n} - 1 \right) + \frac{n}{z} \right).$$

We recall that $\lim_{z \rightarrow \infty} \frac{S+S'}{S} = 1$, $\lim_{k \rightarrow \infty} z_k = \infty$ and $\lim_{k \rightarrow \infty} g(z_k) = 0$. Combined, these facts yield

$$\lim_{k \rightarrow \infty} g'(z_k) = \infty$$

Together with $\lim_{k \rightarrow \infty} g(z_k) = 0$ this implies

$$\lim_{k \rightarrow \infty} g^\#(z_k) = \infty$$

Thus g is not Brody, which in turn implies that f is not Brody. \square

4. CASE STUDY: $e^z + e^{\lambda z}$

Proposition 3. *The entire function $f(z) = e^z + e^{\lambda z}$ is Brody if and only if $\lambda \in \mathbb{R}$.*

Proof. Case 1) The cases $\lambda \in \{0, 1, -1\}$ are trivial.

Case 2) Let $\lambda \notin \mathbb{R}$. Then $f(z) = 0$ iff $e^{z(1-\lambda)} = -1$. Hence the set of all zeros of f is given as the set of all

$$a_k = \frac{(2k+1)\pi i}{1-\lambda}$$

with k running through the set of integers \mathbb{Z} . Now $f(a_k) = 0$ implies $f^\#(a_k) = |f'(a_k)|$ and

$$f'(a_k) = (\lambda - 1)e^{\frac{(2k+1)\pi i \lambda}{1-\lambda}}$$

which is unbounded because $\lambda \notin \mathbb{R}$ implies that

$$\left| e^{\frac{2\pi i \lambda}{1-\lambda}} \right| \neq 1.$$

Therefore $\sup_k f^\#(a_k) = +\infty$ and consequently f is not Brody.

Case 3) Assume $\lambda > 0$. Observe that $f(z) = g(\lambda z)$ for $g(w) = e^w + e^{\frac{1}{\lambda}w}$. Hence we may assume without loss of generality that $0 < \lambda \leq 1$. Since $f(z) = 2e^z$ if $\lambda = 1$, it suffices to consider the case $0 < \lambda < 1$. We choose

$$C = \frac{\log 2}{1-\lambda}.$$

Then

$$|e^{\lambda z}| \leq \frac{1}{2}|e^z|$$

for all $z \in \mathbb{C}$ with $\Re(z) \geq C$. It follows that

$$f^\#(z) = \frac{|e^z + \lambda e^{\lambda z}|}{1 + |e^z + e^{\lambda z}|^2} \leq \frac{3/2|e^z|}{1 + 1/4|e^z|^2} \leq \frac{3/2|e^z|}{1/4|e^z|^2} = 6e^{-\Re z} \leq 6e^{-C}$$

for all $z \in \mathbb{C}$ with $\Re(z) \geq C$.

On the other hand

$$f^\#(z) \leq |f'(z)| \leq e^C + \lambda e^{\lambda C}$$

for all z with $\Re(z) \leq C$. Thus

$$f^\#(z) \leq \max\{6e^{-C}, e^C + \lambda e^{\lambda C}\} \quad \forall z \in \mathbb{C}$$

and consequently f is Brody.

Case 4) We assume $\lambda < 0$. It suffices to consider the case where $-1 < \lambda < 0$.

As before, we define

$$C = \frac{\log 2}{1 - \lambda}.$$

and obtain

$$|e^{\lambda z}| \leq \frac{1}{2}|e^z|$$

and therefore

$$f^\#(z) \leq 6e^{-C}$$

for all $z \in \mathbb{C}$ with $\Re(z) \geq C$.

Next we observe that the condition $\Re(z) \leq -C$ implies

$$|e^z| \leq \frac{1}{2}|e^{\lambda z}|.$$

As a consequence

$$f^\#(z) \leq \frac{3/2|e^{\lambda z}|}{1 + 1/4|e^{\lambda z}|^2} \leq \frac{3/2|e^{\lambda z}|}{1/4|e^{\lambda z}|^2} = 6e^{-\Re \lambda z} \leq 6e^{\lambda C}$$

for all z with $\Re(z) \leq -C$.

Finally we observe that

$$f^\#(z) \leq |f'(z)| \leq |e^z| + |\lambda e^{\lambda z}| \leq e^C + |\lambda|e^{-\lambda C}$$

for all z with $-C \leq \Re(z) \leq C$. □

5. DIVISORS OF SLOW GROWTH

Theorem 1. *Let D be the divisor defined by $D = \{a_k : k \in \mathbb{N}\}$ (all multiplicities being one). Assume that there is a number $\lambda > 1$ such that*

- (1) $|a_{k+1}| > \lambda|a_k| > 0$ for all k , and
- (2) *The origin 0 is contained in the interior of the convex hull of the set of accumulation points of the sequence $\frac{a_k}{|a_k|}$.*

Then there does not exist any Brody function with D as zero divisor.

Proof. The first condition implies the absolute convergence of

$$\sum_k \frac{1}{|a_k|}$$

which implies the convergence of

$$\sum_k \log \left(1 - \frac{1}{|a_k|} \right)$$

which in turn implies the convergence of

$$F(z) = \prod_k \left(1 - \frac{z}{a_k} \right)$$

Thus an arbitrary entire function f with divisor D can be written as

$$f(z) = F(z)e^{g(z)}$$

where F is defined as above while g is an entire function.

Claim. *There is a constant $C > 0$ (depending only on λ) such that*

$$\left| \prod_{k \geq n} \left(1 - \frac{p}{a_k} \right) \right| \geq C$$

for every $n \in \mathbb{N}$ and $p \in \{z \in \mathbb{C} : |p|\sqrt{\lambda} < |a_n|\}$.

To prove the claim, we observe that

$$\left| \prod_{k \geq n} \left(1 - \frac{p}{a_k} \right) \right| \geq C = \prod_{l \geq 0} (1 - \lambda^{-(l+1/2)}).$$

We emphasize that C is independent of n .

We need a second claim.

Claim. *For every $K > 0$ and $m \in \mathbb{N}$ there is a natural number $N \geq m$ (depending on K and m) such that*

$$\left| \prod_{k=m}^n \left(1 - \frac{p}{a_k} \right) \right| \geq K$$

for all $n \geq N$ and p with $|p| \geq \sqrt{\lambda}|a_n|$.

To prove the second claim it suffices to note that

$$\left| \prod_{k=m}^n \left(1 - \frac{p}{a_k} \right) \right| \geq \prod_{l=0}^{n-m} (\lambda^{1/2+l} - 1)$$

and

$$\lim_{s \rightarrow \infty} \prod_{l=0}^s (\lambda^{1/2+l} - 1) = \infty.$$

Next we choose for each $n \in \mathbb{N}$ a complex number p_n such that

$$\sqrt{\lambda}|a_n| < |p_n| < \frac{|a_{n+1}|}{\sqrt{\lambda}}$$

Due to the two above claims we know that there is a number $M \in \mathbb{N}$ such that

$$|F(p_n)| = \left| \prod_{k=1}^{\infty} \left(1 - \frac{p_n}{a_k} \right) \right| \geq 1$$

for all $n \geq M$.

If we set $r_n = \sqrt{|a_{n+1}a_n|}$, we can now deduce:

$$M_{\exp(g)}(r_n) = \max_{|z|=r_n} |g(z)| \leq M_f(r_n).$$

Hence, if f is Brody and therefore $T_f(r) = O(r)$, we can deduce that $T_{\exp(g)}(r) = O(r)$, which in turn implies that g is a polynomial of degree 1, i.e. affine-linear. Hence condition two implies that there is a subsequence a_{n_j} with

$$|\exp(g(a_{n_j}))| \geq 1 \quad \forall j$$

because $|e^w| = e^{\Re(w)}$.

Next we will calculate $|f^\#(a_{n_j})|$. Since $f(a_k) = F(a_k) = 0$ for every k , we have

$$f^\#(a_k) = |f'(a_k)| = |F'(a_k)e^{g(a_k)}| \quad \forall k \in \mathbb{N}$$

and therefore

$$f^\#(a_{n_j}) \geq |F'(a_{n_j})| \quad \forall j \in \mathbb{N}.$$

Thus it suffices to show that

$$\lim_{k \rightarrow \infty} |F'(a_k)| = +\infty$$

in order to deduce that f is not Brody. Now

$$|F'(a_n)| = \frac{1}{|a_n|} \left| \prod_{k=1}^{k \neq n} \left(1 - \frac{a_n}{a_k} \right) \right|.$$

Now

$$\lim_{n \rightarrow \infty} \frac{1}{|a_n|} \left| 1 - \frac{a_n}{a_1} \right| = \frac{1}{|a_1|}.$$

By the first claim

$$\left| \prod_{k > n} \left(1 - \frac{a_n}{a_k} \right) \right| \geq C$$

and by the second claim for each $K > 0$ there is a number $N \in \mathbb{N}$ such that

$$\left| \prod_{1 < k < n} \left(1 - \frac{a_n}{a_k} \right) \right| \geq K$$

for all $n \geq N$.

Combined these assertions show that for each $K > 0$ there is a number N such that

$$|F'(a_n)| \geq \frac{C}{2|a_0|} K$$

for all $n \geq N$. Thus $\limsup |f^\#(a_n)| = +\infty$ and f is not Brody. \square

Corollary 4. *Let $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an unbounded monotone increasing function.*

Then there exists an effective reduced divisor D such that

- (1) *For every $r \in \mathbb{R}^+$ the inequality $\deg(D_r) \leq \zeta(r)$ holds where D_r denotes the restriction of D to the open disc with radius r .*
- (2) *There is no Brody entire function f for which D is the zero divisor.*

In the language of Nevanlinna theory:

Corollary 5. *Let $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an unbounded monotone increasing function. Then there exists an effective reduced divisor D such that*

- (1) *For every $r \in \mathbb{R}^+$ the inequality $N(r, D) \leq \zeta(r)$ holds.*
- (2) *There is no Brody entire function f for which D is the zero divisor.*

6. GROWTH CONDITIONS AND CHARACTERISTIC FUNCTION

Our goal is to show that no bound on the characteristic function $T_f(r)$ forces an entire function f to be Brody except when this bound is strong enough to force f to be a polynomial.

In view of the preceding section the crucial point is to verify that for every such bound there exists an entire function f fulfilling this condition on $T_f(r)$ and fulfilling simultaneously the condition of theorem 1.

Before stating the result of this section we recall some basic notions.

For an entire function f with $f(0) \neq 0$ we may define the characteristic function as $T_f(r) = m_{1/f}(r) + N_{1/f}(r)$ with

$$m_{1/f}(r) = \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})|} d\theta$$

and

$$N_{1/f}(r) = \sum_{a \in \mathbb{C}} \text{ord}_a(f) \log^+ \left| \frac{r}{a} \right|.$$

Now we can state the result:

Theorem 2. *Let $\rho : [1, \infty[\rightarrow]0, \infty[$ be a continuous increasing function.*

- (1) *If $\liminf_{t \rightarrow \infty} \frac{\rho(t)}{\log t} < \infty$, then every entire function with $T_f(r) \leq \rho(r)$ for all $r \geq 1$ must be a polynomial.*
- (2) *If $\liminf_{t \rightarrow \infty} \frac{\rho(t)}{\log t} = \infty$, then there exists an entire function which is not Brody and such that $T_f(r) \leq \rho(r) \forall r \geq 1$.*

Proof. (i). If there is a constant $C > 0$ such that

$$\liminf_{t \rightarrow \infty} \frac{\rho(t)}{\log t} < C$$

then

$$\liminf_{t \rightarrow \infty} \frac{N_{f-a}(t)}{\log t} < C$$

for every entire function f with $T_f(r) \leq \rho(r)$ and for all $a \in \mathbb{C}$. It follows that each fiber $f^{-1}\{a\}$ has cardinality $\leq C$ and that f is a polynomial of degree bounded by C .

(2). As a first preparation, we observe that

$$\prod_k \frac{2^{2k+1} - 1}{2^{2k+1}} = \prod_k \left(1 - \frac{1}{2^{2k+1}}\right) \geq 1 - \sum_k \frac{1}{2^{2k+1}} = 1 - \frac{2}{3} = \frac{1}{3}.$$

We will construct f as follows: We will choose a sequence $c_k \in \mathbb{C}$ with $|c_{k+1}| > 4|c_k| \geq 1$ for all k , then define

$$P_k(z) = 3 \prod_k \left(\frac{z}{c_k} - 1 \right).$$

The conditions on the c_k ensure that P_k converges locally uniformly to an entire function f whose zeroes are precisely the points c_k .

Let $k \in \mathbb{N}$ and let z be a complex number with $\frac{1}{2}|c_{k+1}| \leq |z| \leq 2|c_k|$. Then

$$\left| \frac{z}{c_j} - 1 \right| \geq 1$$

for all $j \leq k$. For $j > k$ then conditions $|c_{l+1}| \geq 4|c_l|$ imply

$$\left| \frac{c_j}{z} \right| \geq 2^{2(j-k)+1}$$

which in turn implies

$$|P_j(z)| \geq 1$$

for all j and all z with $\frac{1}{2}|c_{k+1}| \leq |z| \leq 2|c_k|$. As a consequence,

$$m_{1/f}(r) = 0$$

and hence

$$T_f(r) = N_{1/f}(r)$$

for all $r \in [\frac{1}{2}|c_{k+1}|, 2|c_k|]$ and all $k \in \mathbb{N}$.

Therefore

$$T_f(r) = \sum_{j \leq k} \log \left| \frac{r}{c_j} \right| \leq k \log r$$

for all such r .

Similarly one obtains $T_f(r) = 0$ for $r \leq \frac{1}{2}|c_1|$.

Let us now fix $k \in \mathbb{N}$ and consider $r \in [\frac{1}{2}|c_k|, 2|c_k|]$.

Since $T_f(r)$ is increasing, we have

$$T_f(r) \leq T_f(2|c_k|) = N_{1/f}(2|c_k|) = \sum_{j \leq k} \log \left| \frac{2c_k}{c_j} \right| \leq k \log(2|c_k|) \leq k \log(4r).$$

Summarizing, we have shown that $T_f(r) \leq k \log(4r)$ for all r with $r \leq 2|c_k|$.

Thus it suffices to choose the c_k such that $k \log(4r) \leq \rho(r)$ for all $r \in [2|c_{k-1}|, 2|c_k|]$.

This is possible: We assumed $\liminf_{t \rightarrow \infty} \frac{\rho(t)}{\log t} = \infty$, hence for each $k \in \mathbb{N}$ there is a constant R_k such that $k \log(4r) \leq \rho(r)$ for all $r \geq R_k$. Now it suffices to choose the c_k such that $2|c_{k-1}| \geq R_k$ (in addition to the other conditions $|c_k| \geq 4|c_{k-1}| \geq 1$).

Thus we have established: *We can choose a sequence c_k in $\mathbb{C} \setminus \{0\}$ such that $f(z) = 3 \prod_k \left(\frac{z}{c_k} - 1 \right)$ converges to an entire function f with $T_f(r) \leq \rho(r)$ for all $r \geq 1$.*

Finally, we note that in our construction we choose the c_k such that $|c_{k+1}| \geq 4|c_k|$. Furthermore we may choose the c_k such that the set

$$\left\{ \frac{c_k}{|c_k|} : k \in \mathbb{N} \right\}$$

is dense in $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Then theorem 1 implies such an entire function f is not Brody. \square

7. THE DIVISOR $D = \{k^2 : k \in \mathbb{N}\}$

Proposition 4. *Let $a_k = k^2$. Then there is a Brody function with divisor $\{a_k : k \in \mathbb{N}\}$.*

Proof. We pose

$$f(z) = f_D(z) = \prod_k \left(1 - \frac{z}{k^2} \right)$$

It is easily verified that this is convergent. An explicit calculation shows:

$$f'(a_k) = f'(k^2) = \lim_{n \rightarrow \infty} \frac{1}{2k^2} \frac{(n-1+k) \cdots n}{(n+1) \cdots (n+k)} = \frac{1}{2k^2}.$$

In particular, $|f'(a)| \leq \frac{1}{2}$ for all $a \in D$.

Now let $z \notin |D|$. We will deduce an estimate for

$$\sum_k \frac{1}{|z - a_k|},$$

since

$$\frac{f'(z)}{f(z)} = \sum_k \frac{1}{z - a_k}.$$

First we note: If a subset of \mathbb{C} contains at least m numbers in $|D| = \{k^2 : k \in \mathbb{N}\}$, then its diameter must be at least $m^2 - 1$. Let b_k be a renumbering of the points in $|D|$ such that $|b_{k+1} - z| \geq |b_k - z|$ for all k . If $r = |b_k - z|$ for some $k \in \mathbb{N}$, then

$$b_1, \dots, b_k \in \overline{D_r(z)} = \{w \in \mathbb{C} : |z - w| \leq r\}$$

and therefore $r \geq k^2 - 1$. Thus

$$\frac{1}{|b_k - z|} \leq \frac{1}{k^2 - 1}$$

for all $k > 1$. We deduce: *For every $z \notin |D|$ there is a point $b \in |D|$ with $d(z, D) = |z - b|$ and*

$$\begin{aligned} (1) \quad \left| \left(\sum_k \frac{1}{z - a_k} \right) - \frac{1}{z - b} \right| &= \left| \sum_{k \geq 2} \frac{1}{z - b_k} \right| \leq \sum_{k=2}^{\infty} \frac{1}{k^2 - 1} = \\ &= \frac{1}{2} \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) = \frac{5}{12} \end{aligned}$$

Similarly

$$\sum_{k \in \mathbb{N} \setminus \{m\}} \frac{1}{|a_m - a_k|} \leq \frac{5}{12}$$

for $m \in \mathbb{N}$.

We are now ready to show that $f^\#$ is bounded. We start with the case $d(z, D) \geq 1$. If $d(z, |D|) \geq 1$, then

$$\begin{aligned} f^\#(z) &= \left| \frac{f'(z)}{f(z)} \right| h(f(z)) \\ &\leq (1 - 5/12) h(f(z)) \leq (1 - 5/12) \frac{1}{2} = \frac{7}{24} \end{aligned}$$

It remains to discuss the case $d(z, D) \leq 1$. Let $b \in D$ such that $d(z, D) = d(z, b) = |z - b|$.

As a preparation we discuss

$$P(z) = \prod_{a \in D \setminus \{b\}} \frac{a - z}{a - b} = \prod_{a \in D \setminus \{b\}} \left(1 + \frac{b - z}{a - b} \right).$$

Using $|b - z| \leq 1$ we obtain the following bound:

$$|P(z)| \leq \prod_{a \in D \setminus \{b\}} \left(1 + \frac{1}{|a - b|}\right) \leq \exp \left(\sum_{a \in D \setminus \{b\}} \frac{1}{|a - b|} \right) \leq \exp \left(\frac{5}{12} \right).$$

We know:

$$f'(b) = -\frac{1}{b} \prod_{a \in D \setminus \{b\}} \left(1 - \frac{b}{a}\right)$$

and

$$f(z) = \left(1 - \frac{z}{b}\right) \prod_{a \in D \setminus \{b\}} \left(1 - \frac{z}{a}\right).$$

Combined these two equations yield:

$$\begin{aligned} (z - b) \frac{f'(b)}{f(z)} \prod_{a \in D \setminus \{b\}} \left(1 - \frac{b}{a}\right) &= 1 \\ \iff (z - b) \frac{f'(b)}{f(z)} P(z) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} f'(z) &= \frac{f'(z)}{f(z)} f(z) \\ &= \frac{f'(z)}{f(z)} (z - b) f'(b) P(z) \\ &= \left(\frac{1}{b - z} + \sum_{a \in D \setminus \{b\}} \frac{1}{a - z} \right) (z - b) f'(b) P(z) \\ &= -f'(b) P(z) + \left(\sum_{a \in D \setminus \{b\}} \frac{1}{a - z} \right) (z - b) f'(b) P(z). \end{aligned}$$

It follows that

$$|f'(z)| \leq \frac{1}{2} \exp \left(\frac{5}{12} \right) + \frac{5}{12} 1 \cdot \frac{1}{2} \exp \left(\frac{5}{12} \right)$$

for all $z \in \mathbb{C} \setminus D$ with $d(z, D) \leq 1$. □

8. DISCUSSION

Using the special cases of entire functions studied above we see that the class of entire functions which are Brody is not closed neither under

addition nor under multiplication: the entire functions z , $e^z + 1$ and ze^z are all Brody, but $ze^z + z$ is not, although

$$ze^z + z = z(e^z + 1) = (ze^z) + z.$$

We see also that the Brody condition is neither closed nor open nor complex:

For $(s, t) \in \mathbb{C}^2$ let us consider

$$f_{s,t}(z) = se^z + \frac{z}{tz - 1}$$

By proposition 2 the function $f_{s,t}$ is Brody iff

$$(s, t) \in \{(x, y) \in \mathbb{C}^2 : x = 0 \text{ or } y \neq 0\}$$

which is neither a closed nor an open set.

Moreover, proposition 3 provides an example of a family of entire functions depending holomorphically on a complex parameter λ such that the function is Brody if and only if λ is real.

All this properties are in stark contrast to the situation for Brody curves with values in abelian varieties. If A is an abelian variety, its universal covering is isomorphic to some \mathbb{C}^g . Since every holomorphic map from \mathbb{C} to A lifts to a holomorphic map with values in the universal covering of A , the classical theorem of Liouville implies that an entire curve with values in A is Brody if and only if it can be lifted to an affine-linear map from \mathbb{C} to \mathbb{C}^g . As a consequence one obtains:

- If $f, g : \mathbb{C} \rightarrow A$ are Brody curves, so is $z \mapsto f(z) + g(z)$.
- If $f_t : \mathbb{C} \rightarrow A$ is a family of entire curves depending holomorphically on a parameter $t \in P$ where P is a complex manifold, then the set of all $t \in P$ for which f_t is Brody forms a closed complex analytic subset of P .
- An entire curve $f : \mathbb{C} \rightarrow A$ is Brody if and only if

$$\limsup \frac{\log T_f(r)}{\log r} \leq 2.$$

REFERENCES

- [1] Brody, R.: Compact manifolds and hyperbolicity. *T.A.M.S.* **235**, 213–219 (1978)
- [2] Clunie, J.; Hayman, W.K.: The spherical derivative of integral and meromorphic functions. *Comm. Math. Helv.*, 117–148, **40** (1966)
- [3] Lehto, O.: The spherical derivative of meromorphic functions in the neighbourhood of an isolated singularity. *Comm. Math. Helv.* **33**, 196–205 (1959)
- [4] Lehto, O.; Virtanen, K.I.: On the behaviour of meromorphic functions in the neighbourhood of an isolated singularity. *Ann. Acad. Sci. Fenn. Ser. A. I.*, **240**(1957)

- [5] Tsukamoto, M.: A packing problem for holomorphic curves.
arXiv:math.CV/0605353 (2006)

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